The EPR-B Paradox Resolution. Bell inequalities revisited.

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Abstract. One of the Bell's assumptions in the original derivation of his inequalities was the hypothesis of locality, i.e., the absence of the influence of two remote measuring instruments on one another. That is why violations of these inequalities observed in experiments are often interpreted as a manifestation of the nonlocal nature of quantum mechanics, or a refutation of a local realism. It is well known that the Bell's inequality was derived in its traditional form, without resorting to the hypothesis of locality and without the introduction of hidden variables, the only assumption being that the probability distributions are nonnegative. This can therefore be regarded as a rigorous proof that the hypothesis of locality and the hypothesis of existence of the hidden variables not relevant to violations of Bell's inequalities. The physical meaning of the obtained results is examined. Physical nature of the violation of the Bell inequalities is explained under new EPR-B nonlocality postulate.We show that the correlations of the observables involved in the Bohm–Bell type experiments can be expressed as correlations of classical random variables. The revisited Bell type inequality in canonical notatons reads $\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle \leq 6$.

1. Introduction

One of the Bell's assumptions in the original derivation of his inequalities was the hypothesis of locality, i.e., of the absence of the influence of two remote measuring instruments on one another. That is why violations of these inequalities observed in experiments are often interpreted as a manifestation of the nonlocal nature of quantum mechanics, or a refutation of local realism. However in papers [1], [2], [3] Bell's inequality is derived in its traditional form, without resorting to the hypothesis of locality, the only assumption being that the probability distributions are nonnegative. This can therefore be regarded as a rigorous proof that the hypothesis of locality is not relevant to violations of Bell's inequalities. The physical meaning of the obtained results is examined. In a typical "Bell experiment", two systems which may have previously interacted – for instance they may have been produced by a common source – are now spatially separated and are each measured by one of two distant observers, Alice and Bob. Alice may choose one out of several possible measurements to perform on her system and we let x denote her measurement choice. For instance, x may refer to the position of a knob on her measurement apparatus. Similarly, we let y denote Bob's measurement choice. Once the measurements are performed, they yield outcomes a and b on the two systems.

Remark 1.1. The actual values assigned to the measurement choices x, y and outcomes a, b are purely conventional; they are mere macroscopic labels distinguishing the different possibilities.

Remark 1.2. From one run of the experiment to the other, the outcomes a and b that are obtained may vary, even when the same choices of measurements x and y are made.

Assumption 1.1. These outcomes a and b are thus in general governed by a Kolmogorovian probability distribution $p(ab|xy)$,which can of course depend on the particular experiment being performed. By repeating the experiment a sufficient number of times and collecting the observed data, one can get a fair estimate of such Kolmogorovian probabilities [4], [5].

Assumption 1.2. When such an experiment is actually performed – say, by generating pairs of spin $-1/2$ particles and measuring the spin of each particle in different directions – it will in general be found that

$$p(ab|xy) \neq p(a|x)p(b|y), \tag{1.1}$$

implying that the outcomes on both sides are not statistically independent from each other. Even though the two systems may be separated by a large distance – and may even be space-like separated – the existence of such correlations is nothing mysterious. In particular, it does not necessarily imply some kind of direct influence of one system on the other, for these correlations may simply reveal some dependence relation between the two systems which was established when they interacted in the past. This is at least what one would expect in a local theory. Let us formulate the idea of a local theory more precisely.

Assumption 1.3. The assumption of locality implies that we should be able to identify a set of past factors, described by some variables λ, having a joint causal influence on both outcomes, and which fully account for the dependence between a and b. Once all such factors have been taken into account, the residual indeterminacies about the outcomes must now be decoupled, that is, the Kolmogorovian probabilities for a and b should factorize:

$$p(ab|xy, \lambda) = p(a|x, \lambda)p(b|y, \lambda). \tag{1.2}$$

Remark 1.3. This factorability condition simply expresses that we have found an explanation according to which the probability for a only depends on the past variables λ and on the local measurement x, but not on the distant measurement and outcome, and analogously for the probability to obtain b. The variable λ will not necessarily be constant for all runs of the experiment, even if the procedure which prepares the particles to be measured is held fixed, because λ may involve physical quantities that are not fully controllable. The different values of λ across the runs should thus be characterized by a probability distribution $q(\lambda)$. Combined with the above factorability condition, we can thus write

$$p(ab|xy) = \textstyle\int_\Lambda d\lambda q\,(\lambda)\, p(a|x, \lambda)p(b|y, \lambda), \tag{1.3}$$

where we also implicitly assumed that the measurements x and y can be freely chosen in a way that is independent of λ, i.e., that $q(\lambda|x, y) = q(\lambda)$. This decomposition now represents a precise condition for locality in the context of Bell experiments.

Remark 1.4. Note that no assumptions of determinism or of a "classical behaviour" are being involved in the condition (1.3): we assumed that a (and similarly b) is only probabilistically determined by the measurement x and the variable λ, with no restrictions on the physical laws governing this causal relation. Locality is the crucial assumption behind (1.3). In relativistic terms, it is the requirement that events in one region of space-time should not influence events in space-like separated regions.

Let us consider for simplicity an experiment where there are only two measurement choices per observer $x, y \in \{0, 1\}$ and where the possible outcomes take also two values labelled

$a, b \in \{-1, +1\}$. Let $\langle a_x b_y \rangle$ be the expectation value of the product ab for given measurement choices (x, y) :

$$\langle a_x b_y \rangle = \textstyle\sum_{a,b} abp(ab|xy). \tag{1.4}$$

Consider the following expression

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle\,, \tag{1.5}$$

which is a function of the probabilities $p(ab|xy)$. If these probabilities satisfy the locality decomposition (1.3), we necessarily have that

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \leq 2, \tag{1.6}$$

which is known as the Clauser-Horne-Shimony-Holt (CHSH) inequality [5]. To derive this inequality, we can use (1.3) in the definition (1.4) of $\langle a_x b_y \rangle$, which allows us to express this expectation value as an average

$$\langle a_x b_y \rangle = \textstyle\int_\Lambda d\lambda q\,(\lambda)\, d\lambda q\,(\lambda)\, \langle a_x \rangle_\lambda \langle b_y \rangle_\lambda \tag{1.7}$$

of a product of local expectations

$$\langle a_x \rangle_\lambda = \textstyle\sum_a ap(a|x, \lambda) \tag{1.8}$$

and

$$\langle b_y \rangle_\lambda = \textstyle\sum_b bp(b|y, \lambda) \tag{1.9}$$

taking values in $[-1, 1]$. Inserting this expressions (1.7)-(1.9) in Eq.(1.5), we can write

$$S = \quad \textstyle\int_\Lambda d\lambda q\,(\lambda)\, S_\lambda, \tag{1.10}$$

where

$$S_\lambda = \langle a_0 \rangle_\lambda \langle b_0 \rangle_\lambda + \langle a_0 \rangle_\lambda \langle b_1 \rangle_\lambda + \langle a_1 \rangle_\lambda \langle b_0 \rangle_\lambda - \langle a_1 \rangle_\lambda \langle b_1 \rangle_\lambda\,. \tag{1.11}$$

Since $\langle a_0 \rangle_\lambda, \langle b_0 \rangle_\lambda \in [-1, 1]$, this last expression is smaller than S'_λ

$$S_\lambda \leq S'_\lambda = |\langle b_0 \rangle_\lambda + \langle b_1 \rangle_\lambda| + |\langle b_0 \rangle_\lambda - \langle b_1 \rangle_\lambda|\,. \tag{1.12}$$

Without loss of generality, we can assume that $\langle b_0 \rangle_\lambda \geq \langle b_1 \rangle_\lambda \geq 0$ which yields $S_\lambda \leq 2 \langle b_0 \rangle_\lambda \leq 2$ and thus $S \leq 2$.

Consider now the quantum predictions for an experiment in which the two systems measured by Alice and Bob are two qubits in the singlet state $\Psi^- = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$, where we have used the shortcut notation $|ab\rangle = |a\rangle \otimes |b\rangle$, and where $|0\rangle$ and $|1\rangle$ are conventionally the eigenstates of σ_z for the eigenvalues $+1$ and -1 respectively. Let the measurement choices x and y be associated with vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ corresponding to measurements of $\overrightarrow{x} \cdot \overrightarrow{\sigma}$ on the first qubit and of $\overrightarrow{y} \cdot \overrightarrow{\sigma}$ on the second qubit, where $\overrightarrow{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli vector. According to quantum theory we then have the expectations $\langle a_x b_y \rangle = -\overrightarrow{x} \cdot \overrightarrow{y}$. Let the two settings $x \in \{0, 1\}$ correspond to measurements in the orthogonal directions $\widehat{e}_1$ and $\widehat{e}_2$ respectively and the settings $y \in \{0, 1\}$ to measurements in the directions $-\frac{1}{\sqrt{2}}(\widehat{e}_1 + \widehat{e}_2)$ and $\frac{1}{\sqrt{2}}(-\widehat{e}_1 + \widehat{e}_2)$. We then have $\langle a_0 b_0 \rangle = \langle a_0 b_1 \rangle = \langle a_1 b_0 \rangle = \frac{1}{\sqrt{2}}$ and $-\langle a_1 b_1 \rangle = -\frac{1}{\sqrt{2}}$ whence $S = 2\sqrt{2}$in contradiction with CHSH inequality (1.6).

2. Experimental violation of Bell's inequality under strict Einstein locality conditions. Bell theorem without the hypothesis of locality

Remind that the assumption of locality in the derivation of Bell's theorem requires that the measurement processes of the two observers are space-like separated (Fig.1). This means that it is necessary to freely choose a direction for analysis, to set the analyzer and finally to register the particle such that it is impossible for any information about these processes to travel via any (possibly unknown) channel to the other observer before he, in turn, finishes his measurement. Selection of an analyzer direction has to be completely unpredictable which necessitates a physical random number generator. A numerical pseudo-random number generator can not be used, since its state at any time is predetermined. Furthermore, to achieve complete independence of both observers, one should avoid any common context as would be conventional registration of coincidences as in all previous experiments. Rather the individual events should be registered on both sides completely independently and compared only after the measurements are finished.

This requires independent and highly accurate time bases on both sides. In our experiment for the first time any mutual influence between the two observations is excluded within the realm of Einstein locality. To achieve this condition the observers "Alice" and "Bob" were spatially separated by 400 m across the Innsbruck university science campus.

Remark 2.1. The difference in fiber length was less than $1m$ which means that the photons were registered simultaneously within interval $5ns$.

Figure 1. Spacetime diagram of Bell experiment [6].

Remark 2.2. Assume that photon ν_1 collapses in polarizer **I** at instant t_1 and photon ν_2 collapses in polarizer **II** at instant t_2 respectively. Note that in general case $t_1 \neq t_2$ even if photons ν_1 and ν_2 were registered simultaneously (within interval $5ns$, see Remark 2.1**).**

Notice that obviously there exist only three possibilities:(i) $t_1 - t_2 = 0$,(ii) $t_1 - t_2 = \tau_{\min} = \delta > 0$,(iii) $t_1 - t_2 = -\tau_{\min} = -\delta < 0$. We have choose here $\tau_{\min} = const = \delta = 5ns$.

Remark 2.3. Events where both photomultipliers register a photon within $\Delta t \leq 5ns$ are ignored [6].

Figure 2. A down-converter (one way to produce an entangled pair) throws two entangled photons ν_1 and ν_2 in opposite directions. Polarization of the photons ν_1 and ν_2 measures by polarizers **I** and **II** respectively.
$(i)\mathrm{t}_1 - t_2 = 0, (ii)t_1 - t_2 = \delta > 0, (iii)t_1 - t_2 = -\delta < 0.$

In paper [1], Bell inequality was derived in its traditional form, without resorting to the hypothesis of locality, the only assumption being that the probability distributions are nonnegative. This can therefore be regarded as a rigorous proof that the hypothesis of locality is not relevant to violations of Bell inequalities.

Let A, A', B, B' be random variables with values in the set $\{-1, +1\}$,i.e.,

$$A = \pm 1, A' = \pm 1, B = \pm 1, B' = \pm 1. \tag{2.1}$$

Assume that there exist joint probability distribution functions $W(A, A', B, B')$, of A, A', B, B' defined probabilities for each possible set of outcomes such that:

(i)
$$P(A, A', B, B') \geq 0, P(A, B, B') \geq 0, P(A', B, B') \geq 0, etc., \tag{2.2}$$

(ii)
$$\sum_{A,A',B,B'} P(A, A', B, B') = 1, \quad \sum_{A,A',B,B'} P(A, B, B') = 1, \quad \sum_{A,A',B,B'} P(A', B, B') = 1, etc., \tag{2.3}$$

(iii)
$$\begin{gathered} P(A, A', B, B') + P(-A, A', B, B') = P(A', B, B') \geq P(A, A', B, B'), \\ P(A, A', B, B') + P(A, -A', B, B') = P(A', B, B') \geq P(A, A', B, B'), etc. \end{gathered} \tag{2.4}$$

Note that the following representatons of the quantities $\langle AB \rangle, \langle A'B \rangle, \langle AB' \rangle, \langle A'B' \rangle$ holds [1]

$$\langle AB \rangle = P_{AB}(++) + P_{AB}(--) - P_{AB}(+-) - P_{AB}(-+), etc., \tag{2.5}$$

where

$$P_{AB}(++) = P(A = 1, B = 1), P_{AB}(--) = P(A = -1, B = -1), etc. \tag{2.6}$$

It has been proved that under assumptions (2.1)-(2.4) the following Bell inequality holds [1]

$$|\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle| \leq 2. \tag{2.7}$$

3. The EPR Paradox Resolution.

In order to resolve the EPR Paradox [7] we apply a new quantum mechanical formalism based on the probability representation of continuous observables [8], [9], [10]. We remind that in accordance with a new quantum formalism any given n-dimensional quantum system is identified with a set $\mathbf{Q}$:

$$\mathbf{Q}\left\langle \mathbf{H}, \Im, \Re_{,2,1}, \Im^*(\mathbf{H}), \mathbf{G}, |\psi_t\rangle\right\rangle \tag{3.1}$$

where:

(i) $\mathbf{H}$ that is some infinite-dimensional complex Hilbert space,

(ii) $\Im = (\Omega, , \mathbf{P})$ that is complete probability space,

(iii) $\Re = (R^n, \Sigma)$ that is measurable space,

(iv) $L_{2,1}(\Omega)$ that is complete space of complex valued random variables $X : \Omega \to C^n$ such that

$$_{\Omega}\|X(\omega)\|\, d\mathbf{P} < \infty,_{\Omega}\|X(\omega)\|^2\, d\mathbf{P} < \infty, \tag{3.2}$$

(see [8], Ch. **I**, sec. **II**.1 postulate Q.**I**.1 and [9]).

Remark 3.1. Let $\mathbf{B}_{M_4}$ be a Boolean algebra of physical events in Minkowski spacetime [8] and let $F^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ be measure algebra of physical events in Minkowski spacetime, i.e., $F^{ph}_{M_4}$ that is a Boolean algebra $\mathbf{B}_{M_4}$ with a probability measure $\mathbf{P}$, (see [8], Ch.**I** sec. **III**.2, Def. 3.2.3). We remind that we denote such physical events by $A(\mathbf{x}), B(\mathbf{x}), ...$ etc.,where $\mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ or A, B, ...etc., and we write for a short $A^{Oc}(\mathbf{x}), B^{Oc}(\mathbf{x}), ...$ iff there physical events $A(\mathbf{x}), B(\mathbf{x})$ ... were occured.

Remark 3.2. We assume that particle A is initially in the state $|\psi_\mathbf{A}\rangle \in \mathbf{H}$. Let $A(q,t)\, A\left(|\psi_\mathbf{A}\rangle, \widehat{Q}, q, \delta q, t\right) \in \mathbf{B}_{M_4}$ be a physical event which consists on performing a measurement of the observable $\widehat{Q} = \int_{q_1}^{q_2} q\, |q\rangle\langle q|\, dq$ with an accuracy δq, and the result is obtained in the range $(q - \delta q, q + \delta q)$ at instant t.We assume that $A\left(|\psi_\mathbf{A}\rangle, \widehat{Q}, q, \delta q, t\right) \in^{ph}_{M_4}$.

Remark 3.3. Note that: if there physical event $A\left(|\psi_\mathbf{A}\rangle, \widehat{Q}, q, \delta q, t\right)$ was occurred then immediately after the measurement at instant t unconditional measure $\mathbf{P}$ collapses to conditional measure $\mathbf{P}\left(X|\, A\left(|\psi_\mathbf{A}\rangle, \widehat{Q}, q, \delta q, t\right)\right)$, where $X \in^{ph}_{M_4}$:

$$\mathbf{P}\left(X|\, A\left(|\psi_\mathbf{A}\rangle, \widehat{Q}, q, \delta q, t\right)\right) = \frac{\mathbf{P}\left(X \wedge A\left(|\psi_\mathbf{A}\rangle, \widehat{Q}, q, \delta q, t\right)\right)}{\mathbf{P}\left(A\left(|\psi_\mathbf{A}\rangle, \widehat{Q}, q, \delta q, t\right)\right)}. \tag{3.3}$$

Remark 3.4. Remind if we suppose that a particle at a definite position x is assigned a state vector $|x\rangle \in \mathbf{H}$, and if further we suppose that the possible positions are continuous over the range $(-\infty, \infty)$ and that the associated states are complete, then we are lead to requiring that any state $|\psi_\mathbf{A}\rangle$ of the particle must be expressible as

$$|\psi_\mathbf{A}\rangle = \int_{-\infty}^{\infty} |x\rangle\langle x|\psi_\mathbf{A}\rangle\, dx \tag{3.4}$$

with the states $|x\rangle$ by δ-function normalised, i.e. $\langle x|x'\rangle = \delta(x - x')$.

Definition 3.1. Let $B^\infty = \cup_{[a,b]\subset R}\Sigma_{a,b}$ where $\Sigma_{a,b} = B([a,b])$ is the Borel algebra on a set $[a,b]$. Let $|\psi\rangle \in \mathbf{H}$. We define now a signed measure $\mathbf{P}_{|\psi_\mathbf{A}\rangle} : B^\infty \to R$ by formula

$$\mathbf{P}_{|\psi_\mathbf{A}\rangle}(A) = \int_A x p_{|\psi_\mathbf{A}\rangle}(x)\, d\mu(x), \tag{3.5}$$

where $p_{|\psi_\mathbf{A}\rangle}(x) = |\langle x|\psi_\mathbf{A}\rangle|^2$.

Remark 3.5. We assume now that $(\Omega, , \mathbf{P}) = (R, B^\infty, \mathbf{P}_{B^\infty})$ and $\mathbf{P}_{|\psi\rangle} \ll \mathbf{P}_{\mathbf{B}^\infty}$, i.e. $\mathbf{P}_{|\psi\rangle}$ is absolutely continuous with respect to $\mathbf{P}$. By Radon-Nicodym theorem we obtain for any $A \in \Sigma_{a,b}$:

$$\mathbf{P}_{|\psi\rangle}(A) = \int_A X_{|\psi\rangle}(\omega)\, d\mathbf{P}(\omega), \tag{3.6}$$

i.e.

$$X_{|\psi\rangle}(\omega) = d\mathbf{P}_{|\psi\rangle} d\mathbf{P}. \tag{3.7}$$

Remark 3.6. We assume now that: (i) measure algebra $F_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ admit a representation $\Re[\cdot] : F_{M_4}^{ph} \to (R, B^{\infty}, \mathbf{P}_{B^{\infty}})$ of the measure algebra $F_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ in measure algebra $\mathbf{B}^{\infty} =: (R, B^{\infty}, \mathbf{P}_{B^{\infty}})$, such that

(ii) $\mathbf{P}_{B^{\infty}}(X) = \mathbf{P}(\Re^{-1}[X])$ for any $X \in B^{\infty}$ and

(iii) for any physical event such that $A\left(|\psi_A\rangle, \widehat{Q}, q, \delta q, t\right) \in F_{M_4}^{ph}$ (see Remark 2.6.2) the following condition holds

$$\Re\left[A\left(|\psi_{\mathbf{A}}\rangle, \widehat{Q}, q, \delta q, t\right)\right] = \left\{\omega | q - \delta q \leq X_{|\psi_{\mathbf{A}}\rangle}(\omega) \leq q + \delta q\right\}, \tag{3.8}$$

where $\left\{\omega | q - \delta q < X_{|\psi_{\mathbf{A}}\rangle}(\omega) < q + \delta q\right\} \in \mathbf{B}^{\infty}$.

3.1. The classical weak EPR argument

We briefly remind now the EPR argument [7], [8]. Suppose that a system of two identical particles is prepared in a state such that their relative distance is large and constant $|r_1 - r_2| = L = x_0$, i.e., they are space-like separated, and the total momentum is zero $\overrightarrow{p}_1 + \overrightarrow{p}_2 = 0$ (see Fig. 3.1). This preparation is in principle possible because the two observables say $x_1 - x_2$ and $\overrightarrow{p}_1 + \overrightarrow{p}_2$ are compatible, i.e., both of them can be set to certain values with certainty on the same state. Correspondingly according to quantum mechanics they are in fact represented by commuting operators [8].

Figure 3. Schematic representation of EPR thought experiment.

Remark 3.7. Then one can measure the value of either of the two incompatible single particle observables, say x_1 or p_1 and correspondingly deduce the value of either $x_2 = x_0 - x_1$ or $p_2 = -p_1$ without interacting with particle 2. Because of this they correspond, according to the EPR argument, to elements of reality of the state of particle 2 that are independent of measurements and should be predictable by the theory [7], [8]. On the other hand, quantum mechanics cannot predict the value of both x_2 and p_2 on the same state, because they are incompatible observables and this would be in contrast to Heisenberg uncertainty principle.

Remark 3.8. Thus, conclude EPR, there are elements of reality of a state that cannot be predicted by the theory and therefore the theory is incomplete [7], [8].

Remark 3.9. Note then in additional to canonical EPR thought experiment: (i) one can measure at instant t the value of single particle 1 observable, say x_1^t and deduce the value $x_2^t = x_0 - x_1^t$ of particle 2 at instant t without interacting with particle 2 which at instant t in state say ψ_2^t. Such a measurement however is not disturbs the particle 2 and thus is not alters its state ψ_2^t and therefore the value of single particle 2 observable say p_2 the same as before measurement on particle 1. Therefore one can measure the value p_2^t in state ψ_2^t exactly without any uncertainty. On the other hand, Heisenberg uncertainty principle predicts that the position x_2^t and the momentum p_2^t of any particle cannot both be measured or predict exactly, at the same time t, even in theory.

Let $\mathbf{A}$ and $\mathbf{B}$ be two particles with a state vector $|\psi_{\mathbf{A}}\rangle$

$$|\psi_{\mathbf{A}}\rangle = \int_{-\infty}^{\infty} |x\rangle \langle x|\psi_{\mathbf{A}}\rangle \, dx \tag{3.9}$$

and with a state vector $|\psi_{\mathbf{B}}\rangle$

$$|\psi_{\mathbf{B}}\rangle = \int_{-\infty}^{\infty} |x\rangle \langle x|\psi_{\mathbf{B}}\rangle \, dx \tag{3.10}$$

respectively, and with perfectly correlated position

$$x_{\mathbf{B}} = x_{\mathbf{A}} + x_0 \tag{3.11}$$

and perfectly anti-correlated momentum

$$p_{\mathbf{B}} = -p_{\mathbf{A}}. \tag{3.12}$$

We define now a signed measures $\mathbf{P}_{|\psi_{\mathbf{A}}\rangle} : B^{\infty} \to R$ and $\mathbf{P}_{|\psi_{\mathbf{B}}\rangle} : B^{\infty} \to R$ by formulas

$$\mathbf{P}_{|\psi_{\mathbf{A}}\rangle}(A) = \int_A x p_{|\psi_{\mathbf{A}}\rangle}(x)\, d\mu(x), \tag{3.13}$$

and

$$\mathbf{P}_{|\psi_{\mathbf{B}}\rangle}(A) = \int_A x p_{|\psi_{\mathbf{B}}\rangle}(x)\, d\mu(x), \tag{3.14}$$

where $p_{|\psi_{\mathbf{A}}\rangle}(x) = |\langle x|\psi_{\mathbf{A}}\rangle|^2$ and $p_{|\psi_{\mathbf{B}}\rangle}(x) = |\langle x|\psi_{\mathbf{B}}\rangle|^2$ respectively.

Remark 3.7. We assume now that $(\Omega, , \mathbf{P}) = (R, B^{\infty}, \mathbf{P})$ and (i) $\mathbf{P}_{|\psi_{\mathbf{A}}\rangle} \ll \mathbf{P}$,(ii)$\mathbf{P}_{|\psi_{\mathbf{B}}\rangle} \ll \mathbf{P}$.

We define now random variables $X_{|\psi_{\mathbf{A}}\rangle}(\omega)$ and $X_{|\psi_{\mathbf{B}}\rangle}(\omega)$ by formulas

$$X_{|\psi_{\mathbf{A}}\rangle}(\omega) = d\mathbf{P}_{|\psi_{\mathbf{A}}\rangle} d\mathbf{P}, X_{|\psi_{\mathbf{B}}\rangle}(\omega) = d\mathbf{P}_{|\psi_{\mathbf{B}}\rangle} d\mathbf{P} \tag{3.15}$$

respectively. Notice that from Eq.(3.11), Eq.(3.13)-Eq.(3.14) and Eqs.(3.15) it follows that

$$X_{|\psi_{\mathbf{B}}\rangle}(\omega) = X_{|\psi_{\mathbf{A}}\rangle}(\omega) + x_0, a.s. \tag{3.16}$$

Let $B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right) \in \mathbf{B}_{M_4}$ be a physical event which consists on performing a measurement of the observable $\widehat{X} = \int_{x_1}^{x_2} x\,|x\rangle\langle x|\, dx$ with an accuracy δx, and the result is obtained in the range $(x_{\mathbf{B}} - \delta x, x_{\mathbf{B}} + \delta x)$ at instant t.

Remark 3.8. Note that: if there physical event $B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)$ was occurred then immediately after the measurement at instant t unconditional measure $\mathbf{P}$ collapses to conditional measure $\mathbf{P}\left(X|\, B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right)$, where $X \in F_{M_4}^{ph}$:

$$\mathbf{P}\left(X|\, B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right) = \frac{\mathbf{P}\left(X \wedge B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right)}{\mathbf{P}\left(B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right)}, \tag{3.17}$$

see Remark 3.3.

Notice that: (i) from Eq.(3.8) it follows that

$$\Re\left[B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right] = \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x), \tag{3.18}$$

where we write for short $\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x)$ instead $\left\{\omega | x_{\mathbf{B}} - \delta x \le X_{|\psi_{\mathbf{B}}\rangle}(\omega) \le x_{\mathbf{B}} + \delta x\right\}$,i.e.

$$\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x) \left\{\omega | x_{\mathbf{B}} - \delta x \le X_{|\psi_{\mathbf{B}}\rangle}(\omega) \le x_{\mathbf{B}} + \delta x\right\}, \tag{3.19}$$

see Remark 3.2**,**

(ii) from Eq. (3.11), Eq. (3.16) and Eq. (3.19) it follows that

$$\begin{gathered}\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x) \left\{\omega | x_{\mathbf{B}} - \delta x \le X_{|\psi_{\mathbf{B}}\rangle}(\omega) \le x_{\mathbf{B}} + \delta x\right\} = \\ \left\{\omega |\, (x_{\mathbf{B}} - x_0) - \delta x \le X_{|\psi_{\mathbf{B}}\rangle}(\omega) - x_0 \le (x_{\mathbf{B}} - x_0) + \delta x\right\} = \\ \left\{\omega | x_{\mathbf{A}} - \delta x \le X_{|\psi_{\mathbf{A}}\rangle}(\omega) \le x_{\mathbf{A}} + \delta x\right\} \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{A}}, \delta x),\end{gathered} \tag{3.20}$$

and thus

$$\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x) = \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{A}}, \delta x) \tag{3.21}$$

(iii) from Eq. (3.17)-Eq. (3.19) it follows that: (i) unconditional measure $\mathbf{P}_{B^{\infty}}$ immediately after the measurement at instant t collapses to conditional measure $\mathbf{P}_{B^{\infty}}\left(X|\, \Sigma_{|\psi_{\mathbf{B}}\rangle}(x_{\mathbf{B}}, \delta x)\right)$, where $X \in B^{\infty}$:

$$\mathbf{P}_{B^{\infty}}\left(X|\, \Sigma_{|\psi_{\mathbf{B}}\rangle}(x_{\mathbf{B}}, \delta x)\right) = \frac{\mathbf{P}_{B^{\infty}}\left(X \wedge \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x)\right)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x)\right)}. \tag{3.22}$$

Remark 3.9.(i) From Eq.(3.22) it follows that unconditional probability density function $p_{\mathbf{B}}(x) = |\langle x|\psi_{\mathbf{B}}\rangle|^2$ immediately after the measurement at instant t collapses to the following conditional probability density function as

$$p_{\mathbf{B}}\left(x | \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x)\right) = \begin{matrix} \frac{p_{\mathbf{B}}(x)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x)\right)} & \iff & x \in \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x) \\ 0 & \iff & x \notin \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x), \end{matrix} \tag{3.23}$$

see [8] appendix B.

(ii) From Eq.(3.21) and Eq.(3.22) it follows that unconditional probability density function $p_{\mathbf{A}}(x) = |\langle x|\psi_{\mathbf{A}}\rangle|^2$ immediately after the measurement at instant t collapses to the following conditional probability density function as

$$p_{\mathbf{A}}\left(x|\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right) = \begin{array}{lcl} \frac{p_{\mathbf{A}}(x)}{\mathbf{P}_{B^\infty}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)} & \Longleftrightarrow & x \in \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \\ 0 & \Longleftrightarrow & x \notin \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x). \end{array} \quad (3.24)$$

From Eq.(3.23) it follows that a wave function $\psi_{\mathbf{B}}(x) = \langle x|\psi_{\mathbf{B}}\rangle$ immediately after the measurement at instant t collapses to the following wave function

$$\psi_{\mathbf{B}}^{\mathbf{coll}}(x) = \begin{cases} \frac{\psi_{\mathbf{B}}(x)}{\sqrt{\mathbf{P}_{B^\infty}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right)}} & \Longleftrightarrow \quad x \in \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \\ 0 & \Longleftrightarrow \quad x \notin \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \end{cases} \quad (3.25)$$

From Eq.(3.24) it follows that immediately after the measurement on particle **B** at instant t a wave function $\psi_{\mathbf{A}}(x) = \langle x|\psi_{\mathbf{A}}\rangle$ collapses to the following wave function

$$\psi_{\mathbf{A}}^{\mathbf{coll}}(x) = \begin{cases} \frac{\psi_{\mathbf{A}}(x)}{\sqrt{\mathbf{P}_{B^\infty}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)}} & \Longleftrightarrow \quad x \in \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \\ 0 & \Longleftrightarrow \quad x \notin \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \end{cases} \quad (3.26)$$

Thus measurement on particle **B** alters a wave function $\psi_{\mathbf{A}}(x)$ even if particles **A** and **B** are space-like separatedand and therefore EPR paradox disappears.

4. Bell inequalities revisited

In a typical "Bell experiment", two systems which may have previously interacted – for instance they may have been produced by a common source – are now spatially separated and are each measured by one of two distant observers, Alice and Bob (see Fig.1). Alice may choose one out of several possible measurements to perform on her system and we let x_{t_1} denote her measurement choice at instant t_1. For instance, x_{t_1} may refer to the position of a knob on her measurement apparatus at instant t_1. Similarly, we let y_{t_2} denote Bob's measurement choice. Once the measurements are performed, they yield outcomes a_{t_1} and b_{t_2} on the two systems.

Remark 4.1. The actual values assigned to the measurement choices x_{t_1}, y_{t_2} and outcomes a_{t_1}, b_{t_2} are purely conventional; they are mere macroscopic labels distinguishing the different possibilities.

Remark 4.2. From one run of the experiment to the other, the outcomes a_{t_1} and b_{t_2} that are obtained may vary, even when the same choices of measurements x_{t_1} and y_{t_2} are made.

Remark 4.3. Let $\left\{a_{t_1 x_{t_1}}, b_{t_2 y_{t_2}}\right\}$ be the random event which consists that the definite values of the outcomes a_{t_1} and b_{t_2} were obtained under choices of measurements x_{t_1} and y_{t_2} are made. We denote (i) $\left\{a_{t_1 x_{t_1}}, b_{t_2 y_{t_2}}\right\} := \{a_x b_y\}_=$ if $t_1 = t_2$,(ii) $\left\{a_{t_1 x_{t_1}}, b_{t_2 y_{t_2}}\right\} := \{a_x b_y\}_>$ if $t_1 > t_2$, $\left\{a_{t_1 x_{t_1}}, b_{t_2 y_{t_2}}\right\} := \{a_x b_y\}_<$ if $t_1 < t_2$.

Remark 4.4. We remind that in probability theory, events $E_1, E_2, ..., E_n$ are said to be mutually exclusive if the occurrence of any one of them implies the non-occurrence of the remaining $n-1$ events. Therefore, two mutually exclusive events cannot both occur.

Formally said, the intersection of each two of them is empty (the null event): $E_i \cap E_j = \emptyset$ and therefore $p(E_1 \vee E_2 \vee ... \vee E_n) = \sum_{m=1}^{n} p(E_m)$.

Remark 4.5. Note that events $\Theta_1 = \{a_x b_y\}_=, \Theta_2 = \{a_x b_y\}_>, \Theta_3 = \{a_x b_y\}_<$ obviously are mutually exclusive. Let $\{a_x b_y\}$ be the event $\{a_x b_y\} := \Theta_1 \vee \Theta_2 \vee \Theta_3$ and therefore
$p(\{a_x b_y\}) = p(\Theta_1) + p(\Theta_2) + p(\Theta_3)$.

Remark 4.6. Note that only these events $\{a_x b_y\}$ were really observed in Bell test experiment, see section 2, Remark 2.2.

Assumption 4.1. These outcomes a_{t_1} and b_{t_2} are thus in general governed by a Kolmogorovian probability distribution

$$p(a,t_1;b,t_2|x_{t_1}y_{t_2}), \tag{4.1}$$

which can of course depend on the particular experiment being performed. By repeating the experiment a sufficient number of times and collecting the observed data, one can get a fair estimate of such Kolmogorovian probabilities.

Remark 4.7. Note that in contrast with canonical formula (1.3) Kolmogorovian probability distribution depends on instants t_1 and t_2. These dependencies arise from EPR-B nonlocality postulate (see [8], Ch. I) and also based on accounting of the conditions of the Bell test experiment, see section 2, Remark 2.1-2.2.

Assumption 4.2. The assumption of locality implies that we should be able to identify a set of past factors, described by some variables λ, having a joint causal influence on both outcomes, and which fully account for the dependence between a_{t_1} and b_{t_2}. Once all such factors have been taken into account, the residual indeterminacies about the outcomes must now be decoupled, that is, the Kolmogorovian joint probabilities for a_{t_2} and b_{t_2} should factorize:

$$p(a,t_1;b,t_2|xy,\lambda) = p(a,t_1|x,\lambda)p(b,t_2|y,\lambda). \tag{4.2}$$

Remark 4.8. This factorability condition simply expresses that we have found an explanation according to which the probability for a_{t_1} only depends on the past variables λ and on the local measurement x_{t_1}, but not on the distant measurement and outcome, and analogously for the probability to obtain b_{t_2}.

The variable λ will not necessarily be constant for all runs of the experiment, even if the procedure which prepares the particles to be measured is held fixed, because λ may involve physical quantities that are not fully controllable. The different values of λ across the runs should thus be characterized by a probability distribution $q(\lambda,t_1,t_2)$. Combined with the above factorability condition, we can thus write

$$p(a,t_1;b,t_2|xy) = \textstyle\int_\Lambda d\lambda q\left(\lambda,t_1,t_2\right) p(a,t_1|x,\lambda)p(b,t_2|y,\lambda), \tag{4.3}$$

where we also implicitly assumed that the measurements x_{t_1} and y_{t_2} can be freely chosen in a way that is independent of λ, i.e., that $q(\lambda,t_1,t_2|x_{t_1},y_{t_2}) = q(\lambda,t_1,t_2)$. This decomposition now represents a precise condition for locality in the context of Bell experiments.

Let us consider for simplicity an experiment where there are only two measurement choices per observer $x_{t_1}, y_{t_2} \in \{0,1\}$ and where the possible outcomes take also two values labelled $a_{t_1}, b_{t_2} \in \{-1,+1\}$.

Let $\left(a_{t_1x_{t_1}}, b_{t_2y_{t_2}}\right)$ be the random variable with value of the product $a_{t_1}\cdot b_{t_2}$ for given measurement choices (x_{t_1},y_{t_2}),i.e. $\left(a_{t_1x_{t_1}}, b_{t_2y_{t_2}}\right) = a_{t_1}\cdot b_{t_2}$.

Let $\left\langle\left(a_{t_1x_{t_1}}b_{t_2y_{t_2}}\right)\right\rangle = \left\langle a_{t_1x_{t_1}}\cdot b_{t_2y_{t_2}}\right\rangle$ be the expectation value of the product $a_{t_1}\cdot b_{t_2}$ for given measurement choices (x_{t_1},y_{t_2}) :

$$\left\langle\left(a_{t_1x_{t_1}}, b_{t_2y_{t_2}}\right)\right\rangle = \left\langle a_{t_1x_{t_1}}b_{t_2y_{t_2}}\right\rangle = \textstyle\sum_{a,b} abp(ab,t_1,t_2|x_{t_1}y_{t_2}). \tag{4.4}$$

Remark 4.9. We denote (i) $\left(a_{t_1x_{t_1}}, b_{t_2y_{t_2}}\right) := (a_xb_y)_=$if $t_1 = t_2$,(ii) $\left(a_{t_1x_{t_1}}, b_{t_2y_{t_2}}\right) := (a_xb_y)_>$if $t_1 > t_2$,(iii) $\left(a_{t_1x_{t_1}}, b_{t_2y_{t_2}}\right) := (a_xb_y)_<$if $t_1 < t_2$.

Definition 4.1. Let (a_xb_y) be the random variable defined as follows (i) $(a_xb_y) = (a_xb_y)_=$if $t_1 = t_2$,(ii) $(a_xb_y) = (a_xb_y)_>$ if $t_1 > t_2$, (iii) $(a_xb_y) = (a_xb_y)_<$ if $t_1 < t_2$.

Assumption 4.3. We assume now that $(t_1,t_2) \in Z^2_{+\delta}, Z_{+\delta} = Z_+ \times \delta, 0 < \delta \ll 1$,

$$\begin{array}{c} p(ab,t_1,t_2|x_{t_1}y_{t_2}) = p(ab,t_1-t_2|x_{t_1}y_{t_2}) \iff (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0)\,, \\ q\left(\lambda,t_1,t_2\right) = q\left(\lambda,t_1-t_2\right) \iff (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0)\,. \end{array}. \tag{4.5}$$

Remark 4.10. We denote (i) $p(ab,t_1-t_2|x_{t_1}y_{t_2}) := p_=(ab,t_1-t_2|x_{t_1}y_{t_2})$ if $t_1 = t_2$,
(ii) $p(ab,t_1-t_2|x_{t_1}y_{t_2}) := p_>(ab,t_1-t_2|x_{t_1}y_{t_2})$ if $t_1 > t_2$,
(iii) $p(ab,t_1-t_2|x_{t_1}y_{t_2}) := \quad p_<(ab,t_1-t_2|x_{t_1}y_{t_2})$ if $t_1 < t_2$.

Thus

$$\left\langle \left(a_{t_1 x_{t_1}}, b_{t_2 y_{t_2}} \right) \right\rangle = \left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle = \sum_{a,b} a \cdot b \cdot p(ab, t_1 - t_2 | x_{t_1} y_{t_2})$$
$$and$$
$$\begin{gathered} \langle (a_x b_y) \rangle = \langle a_x b_y \rangle = \sum_{a,b} a \cdot b \cdot p_{=}(ab, t_1 - t_2 | x_{t_1} y_{t_2}) + \\ + \sum_{a,b} a \cdot b \cdot p_{>}(ab, t_1 - t_2 | x_{t_1} y_{t_2}) + \sum_{a,b} a \cdot b \cdot p_{<}(ab, t_1 - t_2 | x_{t_1} y_{t_2}) = \\ \sum_{a,b} a \cdot b \cdot [p_{=}(ab, t_1 - t_2 | x_{t_1} y_{t_2}) + p_{>}(ab, t_1 - t_2 | x_{t_1} y_{t_2}) + \\ p_{<}(ab, t_1 - t_2 | x_{t_1} y_{t_2})] , \end{gathered} \tag{4.6}$$

see Remark 4.5.

Remark 4.11. We denote

$$\left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle =: \langle a_x b_y \rangle_{=} \tag{4.7}$$

iff $|t_1 - t_2| = 0$. We abbreviate

$$\left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle =: \langle a_x b_y \rangle_{>} \tag{4.8}$$

iff $|t_1 - t_2| = \delta$ and $t_1 > t_2$. We abbreviate

$$\left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle =: \langle a_x b_y \rangle_{<} \tag{4.9}$$

iff $|t_1 - t_2| = \delta$ and $t_1 < t_2$. We abbreviate

$$\langle a_x b_y \rangle =: \langle a_x b_y \rangle_{=} + \langle a_x b_y \rangle_{>} + \langle a_x b_y \rangle_{<} . \tag{4.10}$$

Consider the following expression

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle , \tag{4.11}$$

which is a function of the probabilities $p(ab, t_1, t_2 | x_{t_1} y_{t_2})$. If these probabilities satisfy the locality decomposition (4.3) and Eq. (4.5), we necessarily have that

$$S = S_{=} + S_{>} + S_{<} = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \leq 6, \tag{4.12}$$

where

$$\begin{gathered} \langle a_0 b_0 \rangle = \langle a_0 b_0 \rangle_{=} + \langle a_0 b_0 \rangle_{>} + \langle a_0 b_0 \rangle_{<} , \\ \langle a_0 b_1 \rangle = \langle a_0 b_1 \rangle_{=} + \langle a_0 b_1 \rangle_{>} + \langle a_0 b_1 \rangle_{<} , \\ \langle a_1 b_0 \rangle = \langle a_1 b_0 \rangle_{=} + \langle a_1 b_0 \rangle_{>} + \langle a_1 b_0 \rangle_{<} , \\ \langle a_1 b_1 \rangle = \langle a_1 b_1 \rangle_{=} + \langle a_1 b_1 \rangle_{>} + \langle a_1 b_1 \rangle_{<} , \\ and \\ S_{=} = \langle a_0 b_0 \rangle_{=} + \langle a_0 b_1 \rangle_{=} + \langle a_1 b_0 \rangle_{=} - \langle a_1 b_1 \rangle_{=} , \\ S_{>} = \langle a_0 b_0 \rangle_{>} + \langle a_0 b_1 \rangle_{>} + \langle a_1 b_0 \rangle_{>} - \langle a_1 b_1 \rangle_{>} , \\ S_{>} = \langle a_0 b_0 \rangle_{<} + \langle a_0 b_1 \rangle_{<} + \langle a_1 b_0 \rangle_{<} - \langle a_1 b_1 \rangle_{<} . \end{gathered} \tag{4.13}$$

To derive this inequality, we can use (4.3) and Eq. (4.5) in the definitions (4.7)-(4.9) of $\langle a_0 b_0 \rangle_{=}$, $\langle a_x b_y \rangle_{>}$ and $\langle a_x b_y \rangle_{<}$ which allows us to express these expectation values as an averages:

$$\langle a_x b_y \rangle_{=} = \int_\Lambda d\lambda q\left(\lambda, t_1 - t_2\right)(\lambda) \langle a_x \rangle_\lambda^{=} \langle b_y \rangle_\lambda^{=} , \tag{4.14}$$

where $t_1 - t_2 = 0$, and where we abbreviate $\left\langle a_{t_1 x_{t_1}} \right\rangle_\lambda =: \langle a_x \rangle_\lambda^{=}$, $\left\langle b_{t_2 y_{t_2}} \right\rangle_\lambda =: \langle b_y \rangle_\lambda^{=}$,

$$\langle a_x b_y \rangle_{>} = \int_\Lambda d\lambda q\left(\lambda, t_1 - t_2\right)(\lambda) \langle a_x \rangle_\lambda^{>} \langle b_y \rangle_\lambda^{>} , \tag{4.15}$$

where $t_1 > t_2$ and $t_1 - t_2 = \delta$, and where we denote $\left\langle a_{t_1 x_{t_1}} \right\rangle_\lambda =: \langle a_x \rangle_\lambda^{>}$, $\left\langle b_{t_2 y_{t_2}} \right\rangle_\lambda =: \langle b_y \rangle_\lambda^{>}$,

$$\langle a_x b_y \rangle_{<} = \int_\Lambda d\lambda q\left(\lambda, t_1 - t_2\right)(\lambda) \langle a_x \rangle_\lambda^{<} \langle b_y \rangle_\lambda^{<} , \tag{4.16}$$

where $t_1 < t_2$, $t_1 - t_2 = -\delta$, and where we denote $\left\langle a_{t_1 x_{t_1}} \right\rangle_\lambda =: \langle a_x \rangle_\lambda^{<}$, $\left\langle b_{t_2 y_{t_2}} \right\rangle_\lambda =: \langle b_y \rangle_\lambda^{<}$,of a product of corresponding local expectations:

$$\langle a_x \rangle_\lambda^{=} = \sum_a a p(a, t_1 | x, \lambda), \langle b_y \rangle_\lambda^{=} = \sum_b b p(b, t_2 | y, \lambda), \tag{4.17}$$

and

$$\langle a_x \rangle_\lambda^{>} = \sum_a a p(a, t_1 | x, \lambda), \langle b_y \rangle_\lambda^{>} = \sum_b b p(b, t_2 | y, \lambda), \tag{4.18}$$

and

$$\langle a_x \rangle_\lambda^{<} = \sum_a a p(a, t_1 | x, \lambda), \langle b_y \rangle_\lambda^{<} = \sum_b b p(b, t_2 | y, \lambda), \tag{4.19}$$

taking values in $[-1, 1]$. Inserting this expressions (4.17)-(4.19) in Eqs.(4.13), one obtains

$$S_{=} = \int_\Lambda d\lambda q\left(\lambda, 0\right) S_\lambda^{=}, \ S_{>} = \int_\Lambda d\lambda q\left(\lambda, \delta\right) S_\lambda^{>}, \ S_{<} = \int_\Lambda d\lambda q\left(\lambda, -\delta\right) S_\lambda^{<}, \tag{4.20}$$

where

$$
\begin{aligned}
S_{\lambda}^{=} &= \langle a_0\rangle_{\lambda}^{=}\langle b_0\rangle_{\lambda}^{=} + \langle a_0\rangle_{\lambda}^{=}\langle b_1\rangle_{\lambda}^{=} + \langle a_1\rangle_{\lambda}^{=}\langle b_0\rangle_{\lambda}^{=} - \langle a_1\rangle_{\lambda}^{=}\langle b_1\rangle_{\lambda}^{=}\,,\\
S_{\lambda}^{>} &= \langle a_0\rangle_{\lambda}^{>}\langle b_0\rangle_{\lambda}^{>} + \langle a_0\rangle_{\lambda}^{>}\langle b_1\rangle_{\lambda}^{>} + \langle a_1\rangle_{\lambda}^{>}\langle b_0\rangle_{\lambda}^{>} - \langle a_1\rangle_{\lambda}^{>}\langle b_1\rangle_{\lambda}^{>}\,,\\
S_{\lambda}^{<} &= \langle a_0\rangle_{\lambda}^{<}\langle b_0\rangle_{\lambda}^{<} + \langle a_0\rangle_{\lambda}^{<}\langle b_1\rangle_{\lambda}^{<} + \langle a_1\rangle_{\lambda}^{<}\langle b_0\rangle_{\lambda}^{<} - \langle a_1\rangle_{\lambda}^{<}\langle b_1\rangle_{\lambda}^{<}\,.
\end{aligned}
\tag{4.21}
$$

Since $\langle a_0\rangle_{\lambda}, \langle b_0\rangle_{\lambda} \in [-1,1]$, these last expressions is smaller than $\widetilde{S}_{\lambda}^{=}, \widetilde{S}_{\lambda}^{>}$ and $\widetilde{S}_{\lambda}^{<}$ correspondingly, where

$$
\begin{aligned}
S_{\lambda}^{=} \le \widetilde{S}_{\lambda}^{=} &= \left|\langle b_0\rangle_{\lambda}^{=} + \langle b_1\rangle_{\lambda}^{=}\right| + \left|\langle b_0\rangle_{\lambda}^{=} - \langle b_1\rangle_{\lambda}^{=}\right|,\\
S_{\lambda}^{>} \le \widetilde{S}_{\lambda}^{>} &= \left|\langle b_0\rangle_{\lambda}^{>} + \langle b_1\rangle_{\lambda}^{>}\right| + \left|\langle b_0\rangle_{\lambda}^{>} - \langle b_1\rangle_{\lambda}^{>}\right|,\\
S_{\lambda}^{<} \le \widetilde{S}_{\lambda}^{<} &= \left|\langle b_0\rangle_{\lambda}^{<} + \langle b_1\rangle_{\lambda}^{<}\right| + \left|\langle b_0\rangle_{\lambda}^{<} - \langle b_1\rangle_{\lambda}^{<}\right|.
\end{aligned}
\tag{4.22}
$$

Without loss of generality, we can assume that

$$\langle b_0\rangle_{\lambda}^{=} \ge \langle b_1\rangle_{\lambda}^{=} \ge 0,\ \langle b_0\rangle_{\lambda}^{>} \ge \langle b_1\rangle_{\lambda}^{>} \ge 0,\ \langle b_0\rangle_{\lambda}^{<} \ge \langle b_1\rangle_{\lambda}^{<} \ge 0, \tag{4.23}$$

which yields

$$S_{\lambda}^{=} \le 2\langle b_0\rangle_{\lambda}^{=} \le 2, S_{\lambda}^{>} \le 2\langle b_0\rangle_{\lambda}^{>} \le 2, S_{\lambda}^{<} \le 2\langle b_0\rangle_{\lambda}^{<} \le 2 \tag{4.24}$$

and thus

$$S_{=} \le 2, S_{>} \le 2, S_{<} \le 2. \tag{4.25}$$

From (4.25) by summation we obtain

$$S = S_{=} + S_{>} + S_{<} \le 6. \tag{4.26}$$

The inequality (4.26) finalized the proof [8].

5.The canonical Leggett inequality.

Leggett have introduced the class of non-local models and formulated an incompatibility theorem [11]. Such models were extended so as to make it applicable to real experimental situations and also to allow simultaneous tests of all local hidden-variable models. Finally, an experiment was performed that violates the new inequality and hence excludes for the first time a broad class of non-local hidden-variable theories [12]. These theories are based on the following assumptions: (1) all measurement outcomes are determined by pre-existing properties of particles independent of the measurement (realism); (2) physical states are statistical mixtures of subensembles with definite polarization, where (3) polarization is defined such that expectation values taken for each subensemble obey Malus' law (that is, the well-known cosine dependence of the intensity of a polarized beam after an ideal polarizer).

These assumptions are in a way appealing, because they provide a natural explanation of quantum mechanically separable states, e.g., polarization states indeed obey Malus' law. In addition, they do not explicitly demand locality; that is, measurement outcomes may very well depend on parameters in space-like separated regions.

Remark 5.1. Note that assumption (1) requires that an individual binary measurement outcome A for a polarization measurement along direction $\vec{a}$ (that is, whether a single photon is transmitted or absorbed by a polarizer set at a specific angle) is predetermined by some set of hidden-variables λ, and a three-dimensional vector $\vec{u}$, as well as by some set of other possibly non-local parameters η (for example, measurement settings in space-like separated regions) - that is, $A = A(\lambda, \vec{u}, \vec{a}, \eta)$. According to assumption (3), particles with the same $\vec{u}$ but with different λ build up subensembles of 'definite polarization' described by a probability distribution $\rho_{\vec{u}}(\lambda)$. The expectation value $\overline{A}(\vec{u})$, obtained by averaging over λ, fulfils Malus' law, that is, $\overline{A}(\vec{u}) = \int d\lambda \rho_{\vec{u}}(\lambda) A(\lambda, \vec{u}, \vec{a}, \eta) = \vec{u}\cdot\vec{a}$. Finally, with assumption (2), the measured expectation value for a general physical state is given by averaging over the distribution $F(\vec{u})$ of subensembles, that is, $\langle A\rangle = \int d\vec{u} F(\vec{u})\overline{A}(\vec{u})$. Let us consider a specific source, which emits pairs of photons with well-defined polarizations $\vec{u}$ and $\vec{v}$ to laboratories of Alice and Bob, respectively. The local polarization measurement outcomes A and B are fully determined by the polarization vector, by an additional set of hidden variables λ specific to the source and by any set of parameters η outside the source. For reasons of clarity, we choose an explicit non-local dependence of the outcomes on the settings $\vec{a}$ and $\vec{b}$ of the measurement devices. Note, however, that this is just an example of a possible non-local dependence, and that one can choose any other set out of η. Each emitted pair is fully defined by the subensemble distribution $\rho_{\vec{u},\vec{v}}(\lambda)$. In agreement with assumption (3) we impose the following conditions on the predictions for local averages of such measurements (all polarizations and measurement directions are represented as vectors on the Poincaré sphere [10]):

$$\overline{A}(\vec{u}) = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda) A(\vec{a},\vec{b},\lambda) = \vec{u}\cdot\vec{a}, \overline{B}(\vec{v}) = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda) B(\vec{b},\vec{a},\lambda) = \vec{v}\cdot\vec{b}. \tag{5.1}$$

It is important to note that the validity of Malus' law imposes the non-signalling condition on the investigated non-local models, as the local expectation values do only depend on local parameters. The correlation function of measurement results for a source emitting well-polarized photons is defined as the average of the products of the individual measurement outcomes:

$$\overline{AB}(\vec{u},\vec{v}) = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda) A(\vec{a},\vec{b},\lambda) B(\vec{b},\vec{a},\lambda). \tag{5.2}$$

For a general source producing mixtures of polarized photons the observable correlations are averaged over a distribution of the polarizations $F(\vec{u},\vec{v})$, and the general correlation function E is given by:

$$E = \langle AB\rangle = \int d\vec{u} d\vec{v} F(\vec{u},\vec{v}) \overline{AB}(\vec{u},\vec{v}) \tag{5.3}$$

It is a very important trait of this model that there exist subensembles of definite polarizations (independent of measurements) and that the predictions for the subensembles agree with Malus' law. It is clear that other classes of non-local theories, possibly even fully compliant with all quantum mechanical predictions, might exist that do not have this property when reproducing entangled states. There the non-local correlations are a consequence of the non-local quantum potential, which exerts suitable torque on the particles leading to experimental results compliant with quantum mechanics. In that theory, neither of the two particles in a maximally entangled state carries any angular momentum at all when emerging from the source. In contrast, in the Leggett model, it is the total ensemble emitted by the source that carries no angular momentum, which is a consequence of averaging over the individual particles' well defined angular momenta (polarization).

The theories described here are incompatible with quantum theory. Remind the basic idea of the incompatibility theorem [11] uses the following identity, which holds for any numbers $A = \pm 1$ and $B = \pm 1$:

$$-1 + |A + B| = AB = 1 - |A - B|. \tag{5.4}$$

One can apply this identity to the dichotomic measurement results $A = A(\vec{a},\vec{b},\lambda) = \pm 1$ and $B = B(\vec{b},\vec{a},\lambda) = \pm 1$. The identity holds even if the values of A and B mutually depend on each other. For example, the value of a specific outcome A can depend on the value of an actually obtained result B. In contrast, in the derivation of the CHSH inequality it is necessary to assume that A and B do not depend on each other. Therefore, any kind of non-local dependencies used in the present class of theories are allowed. Taking the average over the subensembles with definite polarizations we obtain:

$$-1 + \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda)|A + B| = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda) AB = 1 - \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda)|A - B| \tag{5.5}$$

Denoting these averages by $\langle\cdot\rangle$, one arrives at the shorter expression:

$$-1 + \langle|A + B|\rangle = \langle AB\rangle = 1 - \langle|A - B|\rangle\,. \tag{5.6}$$

As the average of the modulus is greater than or equal to the modulus of the averages, we get the inequalities:

$$-1 + |\,\langle A\rangle + \langle B\rangle\,| \leq \langle AB\rangle \leq 1 - |\,\langle A\rangle - \langle B\rangle\,|. \tag{5.7}$$

By inserting Malus' law, equations (5.1), in equation (5.7), and by using expression (5.3), one gets at a set of inequalities for experimentally accessible correlation functions [12]. In particular, if we let Alice choose her observable from the set of two settings $\vec{a}_1$ and $\vec{a}_2$, and Bob from the set of three settings $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3 = \vec{a}_2$, the following generalized Leggett-type inequality is obtained [12]:

$$S_{NLHV} = |E_{11}(\varphi) + E_{23}(0)| + |E_{22}(\varphi) + E_{23}(0)| \leq 4 - \tfrac{4}{\pi}|\sin\tfrac{\varphi}{2}|, \quad (5.8)$$

where $E_{kl}(\varphi)$ is a uniform average of all correlation functions, defined in the plane of $\vec{a}_k$ and $\vec{b}_l$, with the same relative angle φ; the subscript NLHV stands for 'non-local hidden- variables'. For the inequality to be applied, vectors $\vec{a}_1$ and $\vec{b}_1$ necessarily have to lie in a plane orthogonal to the one defined by $\vec{a}_2$ and $\vec{b}_2$. This contrasts with the standard experimental configuration used to test the CHSH inequality, which is maximally violated for settings in one plane.

Figure 4. Testing non-local hidden-variable theories [12].

It is well known that quantum theory violates inequality (5.8). Consider the quantum predictions for the polarization singlet state of two photons, $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}[|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B]$, where, for example, $|H\rangle_A$ denotes a horizontally polarized photon propagating to Alice. The quantum correlation function for the measurements $\vec{a}_k$ and $\vec{b}_l$ performed on photons depends only on the relative angles between these vectors, and therefore $E_{kl} = -\vec{a}_k \cdot \vec{b}_l = -\cos\varphi$. Thus the left hand side of inequality (5.8), for quantum predictions, reads $|2(\cos\varphi + 1)|$. The maximal violation of inequality (5.8) is for $\varphi_{max} = 18.8°$. For this difference angle, the bound given by inequality (5.8) equals 3.792 and the quantum value is 3.893. Although this excludes canonical the non-local models, it might still be possible that the obtained correlations could be explained by a local realistic model. In order to avoid that, we have to exclude both local realistic and non-local realistic hidden-variable theories. Note however that such local realistic theories need not be constrained by assumptions (1)-(3). The violation of the CHSH inequality invalidates the all canonical local realistic models. If one takes

$$S_{CHSH} = |E_{11} + E_{12} - E_{21} + E_{22}| \leq 2 \quad (5.9)$$

the quantum value of the left hand side for the settings used to maximally violate inequality (5.8) is 2.2156, see [12]. The correlation function determined in an actual experiment is typically reduced by a visibility factor V to $E^{exp} = -V\cos\varphi$ owing to noise and imperfections. Thus to observe violations of inequality (5.8) (and inequality (5.9)) in the experiment, one must have a sufficiently high experimental visibility of the observed interference. For the optimal difference angle $\varphi_{max} = 18.8°$, the minimum required visibility is given by the ratio of the bound (3.792) and the quantum value (3.893) of inequality (5.8), or $\sim$ 97.4%. We note that in standard Bell-type experiments, a minimum visibility of only $\sim$ 71% is sufficient to violate the CHSH inequality, inequality (5.9), at the optimal settings. For the settings used here, the critical visibility reads $2/2.2156 \approx 90.3\%$, which is much lower than 97.4%.

Figure 5. Experimental set-up [12].

A 2-mm-thick type-II -barium-borate (BBO) crystal is pumped with a pulsed frequency-doubled Ti-sapphire laser (180 fs) at λ= 395 nm wavelength and ˜150 mW optical c.w. power. The crystal is aligned to produce the polarization-entangled singlet state $|\Psi^-\rangle_{AB} =$

$\frac{1}{\sqrt{2}}[|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B]$. Spatial and temporal distinguishability of the produced photons (induced by birefringence in the BBO) are compensated by a combination of half-wave plates ($\lambda/2$) and additional BBO crystals (BBO/2), while spectral distinguishability (due to the broad spectrum of the pulsed pump) is eliminated by narrow spectral ltering of 1 nm bandwidth in front of each detector.

In addition, the reduced pump power diminishes higher-order SPDC emissions of multiple photon pairs. This allows us to achieve a two-photon visibility of about 99%, which is well beyond the required threshold of 97:4%. The arrows in the Poincare spheres indicate the measurement settings of Alice's and Bob's polarizers for the maximal violation of inequality (3.3.9). Note that setting $\overrightarrow{b}_2$ lies in the $y - z$ plane and therefore a quarter-wave plate has to be introduced on Bob's side. The coloured planes indicate the measurement directions for various difference angles φ for both inequalities.

In terms of experimental count rates, the correlation function $E(\vec{a}, \vec{b})$ for a given pair of general measurement settings is defined by

$$E(\vec{a}, \vec{b}) = \frac{N_{++} + N_{--} - N_{+-} - N_{-+}}{N_{++} + N_{--} + N_{+-} + N_{-+}}, \tag{5.10}$$

where N_{AB} denotes the number of coincident detection events between Alice's and Bob's measurements within the integration time. We ascribe the number +1, if Alice (Bob) detects a photon polarized along $\vec{a}$ $\left(\vec{b}\right)$, and -1 for the orthogonal direction $\vec{a}^{\perp}$ $\left(\vec{b}^{\perp}\right)$. For example, N_{+-} denotes the number of coincidences in which Alice obtains $\vec{a}$ and Bob $\vec{b}^{\perp}$. Note that $E(\vec{a}_k, \vec{b}_l) = E_{kl}(\varphi)$, where φ is the difference angle between the vectors $\vec{a}$ and $\vec{b}$ on the Poincaré sphere. To test inequality (5.8), three correlation functions ($E(\vec{a}_1, \vec{b}_1,), E(\vec{a}_2, \vec{b}_2), E(\vec{a}_2, \vec{b}_3)$) have to be extracted from the measured data. We choose observables $\vec{a}_1$ and $\vec{b}_1$ as linear polarization measurements (in the x-z plane on the Poincaré sphere; see Fig. 5) and $\vec{a}_2$ and $\vec{b}_2$ as elliptical polarization measurements in the y-z plane. Two further correlation functions ($E(\vec{a}_2, \vec{b}_1)$ and $E(\vec{a}_1, \vec{b}_2)$) are extracted to test the CHSH inequality (5.9).

Figure 6 shows the experimental violation of inequalities (5.8) and (5.9) for various difference angles. Maximum violation of inequality (5.8) is achieved, for example, for the settings $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3\} = \{45°, 0°, 55°, 10°, 0°\}$.

Figure 6. Experimental violation of the inequalities for non-local hidden-variable theories (NLHV) and for local realistic theories (CHSH) [12].

(**a**) Dashed line indicates the bound of inequality (5.8) for the investigated class of nonlocal hiddenvariable theories. The solid line is the quantum theoretical prediction reduced by the experimental visibility. The shown experimental data were taken for various difference angles φ (on the Poincaré sphere) of local measurement settings. The bound is clearly violated for $4° < \varphi < 36°$. Maximum violation is observed for $\varphi_{max} = 20°$.

(**b**) At the same time, no local realistic theory can model the correlations for the investigated settings as the same set of data also violates the CHSH inequality (5.9). The bound (dashed line) is overcome for all values φ around φ_{max}, and hence excludes any local realistic explanation of the observed correlations in **a**.

6.Leggett inequality revisited. Validity of the revised Leggett inequality.

In [10] we have introduced the new class of non-local models. Such models were extended so as to make it applicable to real experimental situations and also to allow simultaneous tests of all local hidden-variable models considered in sect 4. A general framework of such nonclassical models is the following: assumption (1) requires that an individual binary measurement outcome A_{t_1} for a polarization measurement at instant t_1 along direction $\vec{a}$ (that is, whether a single photon is transmitted or absorbed at instant t_1 by a polarizer set at a specific angle) is predetermined by some set of hidden-variables λ, and a three-dimensional vector $\vec{u}$, as well as by some set of other possibly non-local parameters η (for example, measurement settings in space-like separated regions) - that is, $A_{t_1}(\lambda, \vec{u}, \vec{a}, \eta) = A(\lambda, \vec{u}, \vec{a}, \eta, t_1)$. According to assumption (3), particles with the same $\vec{u}$ but with different λ build up subensembles of 'definite polarization' described by a probability distribution $\rho_{\vec{u}}(\lambda, t_1)$. The expectation value $\langle A_{t_1}(\vec{u})\rangle = \langle A(\vec{u}, t_1)\rangle$, obtained by averaging over λ, fulfils Malus' law, that is, $\langle A_{t_1}(\vec{u})\rangle = \int d\lambda \rho_{\vec{u}}(\lambda, t_1) A(\lambda, \vec{u}, \vec{a}, \eta, t_1) = \vec{u}\cdot\vec{a}$. Finally, with assumption (2), the measured expectation value for a general physical state is given by averaging over the distribution $F(\vec{u}, t_1)$ of subensembles, that is, $\langle A_{t_1}\rangle = \int d\vec{u} F(\vec{u}, t_1)\langle A_{t_1}(\vec{u})\rangle$. Let us consider a specific source, which emits pairs of photons with well-defined polarizations $\vec{u}$ and $\vec{v}$ to laboratories of Alice and Bob, respectively. The local polarization measurement outcomes A_{t_1} and B_{t_2} are fully determined by the polarization vector, by an additional set of hidden variables λ specific to the source and by any set of parameters η outside the source. For reasons of clarity, we choose an explicit non-local dependence of the outcomes on the settings $\vec{a}$ and $\vec{b}$ of the measurement devices. Note, however, that this is just an example of a possible non-local dependence, and that one can choose any other set out of η. Each emitted pair is fully defined by the subensemble distribution $\rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2)$. In agreement with assumption (3) we impose the following conditions on the predictions for local averages of such measurements (all polarizations and measurement directions are represented as vectors on the Poincaré sphere):

$$\langle A_{t_1}(\vec{u})\rangle = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) A(\vec{a}, \vec{b}, \lambda, t_1) = \vec{u}\cdot\vec{a}, \tag{6.1}$$

$$\langle B_{t_2}(\vec{v})\rangle = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) B(\vec{b}, \vec{a}, \lambda, t_2) = \vec{v}\cdot\vec{b}. \tag{6.2}$$

It is important to note that the validity of Malus' law imposes the non-signalling condition on the investigated non-local models, as the local expectation values do only depend on local parameters. The correlation function of measurement results for a source emitting well-polarized photons is defined as the average of the products of the individual measurement outcomes:

$$\langle A_{t_1} B_{t_2}(\vec{u}, \vec{v})\rangle = \langle AB(\vec{u}, \vec{v}, t_1, t_2)\rangle = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) A(\vec{a}, \vec{b}, \lambda, t_1) B(\vec{b}, \vec{a}, \lambda, t_2) \tag{6.3}$$

For a general source producing mixtures of polarized photons the observable correlations are averaged over a distribution of the polarizations $F(\vec{u}, \vec{v}, t_1, t_2)$, and the general correlation function $E(t_1, t_2)$ is given by:

$$E(t_1, t_2) = \langle A_{t_1} B_{t_2}\rangle_F = \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}, t_1, t_2)\langle AB(\vec{u}, \vec{v}, t_1, t_2)\rangle . \tag{6.4}$$

Assumption 6.1. We assume now that $|t_1 - t_2| = \delta > 0$, and

$$\begin{gathered} \langle AB(\vec{u}, \vec{v}, t_1, t_2)\rangle = \langle AB(\vec{u}, \vec{v}, t_1 - t_2)\rangle , \\ F(\vec{u}, \vec{v}, t_1, t_2) = F(\vec{u}, \vec{v}, t_1 - t_2), \\ E(t_1, t_2) = E(t_1 - t_2) . \end{gathered} \tag{6.5}$$

Remark 6.1. We abbreviate now for short

$$\begin{gathered} \langle A_{t_1} B_{t_2}(\vec{u}, \vec{v})\rangle \, \langle AB(\vec{u}, \vec{v})\rangle^{>} \, iff t_1 - t_2 = \delta > 0, \\ \langle A_{t_1} B_{t_2}(\vec{u}, \vec{v})\rangle \, \langle AB(\vec{u}, \vec{v})\rangle^{<} \, iff t_1 - t_2 = -\delta < 0, \\ E(t_1, t_2) \, E^{>} iff t_1 - t_2 = \delta > 0, E(t_1, t_2) \, E^{<} iff t_1 - t_2 = -\delta < 0. \end{gathered} \tag{6.6}$$

We take a source which distributes pairs of well-polarized photons. Different pairs can have different polarizations. The size of a subensemble in which photons have polarizations $\vec{u}$ and $\vec{v}$ is described by the weight function $F(\vec{u}, \vec{v}, t_1, t_2) = F(\vec{u}, \vec{v}, t_1 - t_2)$. All polarizations and measurement directions are represented as vectors on the Poincaré sphere. In every such subensemble individual measurement outcomes are determined by hidden variables λ. The hidden variables are distributed according to the distribution $\rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) = \rho_{\vec{u},\vec{v}}(\lambda, t_1 - t_2)$. For any dichotomic measurement results, $A_{t_1} = \pm 1$ and $B_{t_2} = \pm 1$, the following identity holds:

$$-1 + |A_{t_1} + B_{t_2}| = A_{t_1} B_{t_2} = 1 - |A_{t_1} - B_{t_2}|. \tag{6.7}$$

If the signs of A_{t_1} and B_{t_2} are the same $|A_{t_1} + B_{t_2}| = 2$ and $|A_{t_1} - B_{t_2}| = 0$, and if $A_{t_1} = -B_{t_2}$ then $|A_{t_1} + B_{t_2}| = 0$ and $|A_{t_1} - B_{t_2}| = 2$. Any kind of non-local dependencies is allowed, i.e. $A_{t_1} = A(\vec{a}, \vec{b}, \vec{u}, \vec{v}, \lambda, t_1 ....)$ and $B_{t_2} = B(\vec{a}, \vec{b}, \vec{u}, \vec{v}, \lambda, t_2 ....)$. Taking ther average over the subensemble with definite polarizations gives

$$\begin{gathered} -1 + \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) |A_{t_1} + B_{t_2}| = \\ \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) A_{t_1} B_{t_2} = 1 - \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) |A_{t_1} - B_{t_2}|, \end{gathered} \tag{6.8}$$

which in an abbreviated notation, where the averages are denoted by $\langle \cdot \rangle$, is

$$-1 + \langle |A_{t_1} + B_{t_2}| \rangle = \langle A_{t_1} B_{t_2} \rangle = 1 - \langle |A_{t_1} - B_{t_2}| \rangle . \tag{6.9}$$

As the average of the modulus is greater than or equal to the modulus of the averages, we get the inequalities:

$$-1 + |\langle A_{t_1} \rangle + \langle B_{t_2} \rangle| \leq \langle A_{t_1} B_{t_2} \rangle \leq 1 - |\langle A_{t_1} \rangle - \langle B_{t_2} \rangle|. \tag{6.10}$$

From now on only the upper bound will be considered, however all steps apply to the lower bound as well. With the assumption that photons with well defined polarization obey Malus' law

$$\langle A_{t_1} \rangle = \vec{u} \cdot \vec{a}, \ \langle B_{t_2} \rangle = \vec{v} \cdot \vec{b}, \tag{6.11}$$

the upper bound of the inequality (6.10) becomes

$$\langle A_{t_1} B_{t_2} \rangle \leq 1 - |\vec{u} \cdot \vec{a}_k - \vec{v} \cdot \vec{b}_l|, \tag{6.12}$$

where $\vec{a}_k$ and $\vec{b}_l$ are unit vectors associated with the k-th measurement setting of Alice and the l-th of Bob, respectively. Taking the average over arbitrary polarizations we obtain

$$E_{kl}(t_1, t_2) \leq 1 - \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}, t_1, t_2) |\vec{u} \cdot \vec{a}_k - \vec{v} \cdot \vec{b}_l|. \tag{6.13}$$

Remark 6.2. We assume now that $t_1 - t_2 = \delta > 0$ and by using (6.5)-(6.6) we rewrite now the inequalities (6.12) and (6.13) as

$$\langle AB \rangle^{>} \leq 1 - |\vec{u} \cdot \vec{a}_k - \vec{v} \cdot \vec{b}_l| \tag{6.14}$$

and

$$E_{kl}^{>} \leq 1 - \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}, \delta) |\vec{u} \cdot \vec{a}_k - \vec{v} \cdot \vec{b}_l|. \tag{6.15}$$

respectively.

Remark 6.3. We assume now that $t_1 - t_2 = -\delta < 0$ and by using (6.5)-(6.6) we rewrite now the inequalities (6.12) and (6.13) as

$$\langle AB \rangle^{<} \leq 1 - |\vec{u} \cdot \vec{a}_k - \vec{v} \cdot \vec{b}_l| \tag{6.16}$$

and

$$E_{kl}^{<} \leq 1 - \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}, -\delta) |\vec{u} \cdot \vec{a}_k - \vec{v} \cdot \vec{b}_l|. \tag{6.17}$$

respectively.

Remark 6.4. We define now the full avereges E_{kl}^{GHVT} as

$$E_{kl}^{GHVT} = E_{kl}^{>} + E_{kl}^{<}, \tag{6.18}$$

where E_{kl}^{GHVT} is the correlation function which can be experimentally measured when Alice chooses to measure $\vec{a}_k$ and Bob chooses $\vec{b}_l$. Let us denote by u_{kl} and v_{kl} the length of projections of vectors $\vec{u}$ and $\vec{v}$ onto the plane spanned by $\vec{a}_k$ and $\vec{b}_l$.

Thus from the inequalities (6.15) and (6.17) by using Eq.(6.18) we obtain

$$\begin{gathered} E_{kl}^{GHVT} \leq 2 - \int d\vec{u} d\vec{v} \widetilde{F}(\vec{u}, \vec{v}) |\vec{u} \cdot \vec{a}_k - \vec{v} \cdot \vec{b}_l|, \\ \widetilde{F}(\vec{u}, \vec{v}, \delta) = F(\vec{u}, \vec{v}, \delta) + F(\vec{u}, \vec{v}, -\delta). \end{gathered} \tag{6.19}$$

Let us denote by u_{kl} and v_{kl} the length of projections of vectors $\vec{u}$ and $\vec{v}$ onto the plane spanned by $\vec{a}_k$ and $\vec{b}_l$. Since one can decompose vectors $\vec{u}$ and $\vec{v}$ into a vector orthogonal to the plane of the settings and a vector within the plane the scalar products read

$$\vec{u}\cdot\vec{a}_k = u_{kl}\cos(\phi_{a_k}-\phi_u), \vec{v}\cdot\vec{b}_l = v_{kl}\cos(\phi_{b_l}-\phi_v), \tag{6.20}$$

where all ϕ angles are relative to some axis within the plane of the settings; Angles ϕ_u and ϕ_v describe the position of the projections of vectors $\vec{u}$ and $\vec{v}$, respectively, whereas angles ϕ_{a_k} and ϕ_{b_l} describe the position of the setting vectors. With this notation the inequality (6.19) becomes to

$$E_{kl}^{GHVT} \leq 2 - \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)|u_{kl}\cos(\phi_{a_k}-\phi_u) - v_{kl}\cos(\phi_{b_l}-\phi_v)|. \tag{6.21}$$

The magnitudes of the projections can always be decomposed into the sum and the difference of two real numbers $u_{kl} = n_1 + n_2$ and $v_{kl} = n_1 - n_2$. We insert this decomposition into the last inequality, and hence the terms multiplied by n_1 and n_2 are

$$\begin{gathered}\cos(\phi_{a_k}-\phi_u) - \cos(\phi_{b_l}-\phi_v) = \\ = 2\sin\frac{\phi_{a_k}+\phi_{b_l}-(\phi_u+\phi_v)}{2}\sin\frac{-(\phi_{a_k}-\phi_{b_l})+\phi_u-\phi_v}{2}\end{gathered} \tag{6.22}$$

and

$$\begin{gathered}\cos(\phi_{a_k}-\phi_u) - \cos(\phi_{b_l}-\phi_v) = \\ = 2\cos\frac{\phi_{a_k}+\phi_{b_l}-(\phi_u+\phi_v)}{2}\cos\frac{\phi_{a_k}-\phi_{b_l}-(\phi_u-\phi_v)}{2},\end{gathered} \tag{6.23}$$

respectively. We make the following substitution for the measurement angles

$$\xi_{kl} = \frac{\phi_{a_k}+\phi_{b_l}}{2}, \varphi_{kl} = \phi_{a_k} - \phi_{b_l}, \tag{6.24}$$

and parameterize the position of the projections within their plane by

$$\psi_{uv} = \frac{\phi_u+\phi_v}{2}, \chi_{uv} = \phi_u - \phi_v. \tag{6.25}$$

Using these new angles one obtaines that

$$\begin{gathered}E_{kl}^{GHVT}(\xi_{kl},\varphi_{kl}) \leq 2- \\ 2\int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)|n_2\cos\frac{\varphi_{kl}-\chi_{uv}}{2}\cos(\xi_{kl}-\psi_{uv}) - n_1\sin\frac{\varphi_{kl}-\chi_{uv}}{2}\sin(\xi_{kl}-\psi_{uv})|,\end{gathered} \tag{6.26}$$

where in the correlation function $E_{kl}(\xi_{kl},\varphi_{kl})$ we explicitly state the angles it is dependent on. The expression within the modulus is a linear combination of two harmonic functions of $\xi_{kl}-\psi_{uv}$, and therefore is a harmonic function itself. Its amplitude reads

$$\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)}, \tag{6.27}$$

and the phase is some fixed real number α

$$\begin{gathered}E_{kl}^{GHVT}(\xi_{kl},\varphi_{kl}) \leq \\ 2 - 2\int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)}\times \\ \times|\cos(\xi_{kl}-\psi_{uv}+\alpha)|.\end{gathered} \tag{6.28}$$

In the next step we average both sides of this inequality over the measurement angle $\xi_{kl} = \frac{\phi_{a_k}+\phi_{b_l}}{2}$. This means an integration over $\xi_{kl} \in [0, 2\pi)$ and a multiplication by $\frac{1}{2\pi}$. The integral of the ξ_{kl} dependent part of the right-hand side of (6.28) reads:

$$\frac{1}{2\pi}\int_0^{2\pi} d\xi_{kl}\,|\cos(\xi_{kl}-\psi_{uv}+\alpha)| = \frac{2}{\pi}. \tag{6.29}$$

By denoting the average of the correlation function over the angle ξ_{kl} as:

$$E_{kl}^{GHVT}(\varphi_{kl}) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\xi_{kl}E_{kl}^{GHVT}(\xi_{kl},\varphi_{kl}), \tag{6.30}$$

one can write (6.28) as

$$E_{kl}^{GHVT}(\varphi_{kl}) \leq 2 - \frac{4}{\pi}\int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)} \tag{6.31}$$

This inequality is valid for any choice of observables in the plane defined by $\vec{a}_k$ and $\vec{b}_l$. One can introduce two new observable vectors in this plane and write the inequality for the averaged correlation function $E_{k'l'}^{GHVT}(\varphi'_{k'l'})$ of these new observables. The sum of these two inequalities is

$$\begin{gathered}E_{kl}^{GHVT}(\varphi_{kl}) + E_{k'l'}^{HVT}(\varphi'_{k'l'}) \leq 4 - \frac{4}{\pi}\int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta) \\ \times\left(\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)} + \right. \\ \left. + \sqrt{n_2^2\cos^2\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right)}\right).\end{gathered} \tag{6.32}$$

One can use the triangle inequality

$$\begin{gathered} ||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||, \\ \sqrt{(x_1+y_1)^2+(x_2+y_2)^2} \leq \sqrt{x_1^2+x_2^2}+\sqrt{y_1^2+y_2^2}, \end{gathered} \tag{6.33}$$

for the two-dimensional vectors $\vec{x}=(x_1,x_2)$ and $\vec{y}=(y_1,y_2)$, with components defined by

$$x_1 = \left|n_2 \cos\frac{\varphi_{kl}-\chi_{uv}}{2}\right|, \quad y_1 = \left|n_2 \cos\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right|, \tag{6.34}$$

and
$$x_2 = \left|n_1 \sin\frac{\varphi_{kl}-\chi_{uv}}{2}\right|, y_2 = \left|n_1 \sin\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right|. \tag{6.35}$$

One can further estimate this bound by using the following relations

$$\left|\cos\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)\right| + \left|\cos\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right)\right| \geq \left|\sin\left(\frac{\varphi_{kl}-\varphi'_{k'l'}}{2}\right)\right| \tag{6.36}$$

and
$$\left|\sin\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)\right| + \left|\sin\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right)\right| \geq \left|\sin\left(\frac{\varphi_{kl}-\varphi'_{k'l'}}{2}\right)\right|. \tag{6.37}$$

This estimate follows if one uses the formula for the sine of the difference angle to the right-hand side argument $\frac{\varphi_{kl}-\varphi'_{k'l'}}{2} = \frac{\varphi_{kl}-\chi_{uv}}{2} - \frac{\varphi'_{k'l'}-\chi_{uv}}{2}$. Namely,

$$\begin{aligned} \left|\sin\frac{\varphi_{kl}-\varphi'_{k'l'}}{2}\right| &= \left|\sin\frac{\varphi_{kl}-\chi_{uv}}{2}\cos\frac{\varphi'_{k'l'}-\chi_{uv}}{2} - \cos\frac{\varphi_{kl}-\chi_{uv}}{2}\sin\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right| \\ &\leq \left|\sin\frac{\varphi_{kl}-\chi_{uv}}{2}\right|\left|\cos\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right| + \left|\cos\frac{\varphi_{kl}-\chi_{uv}}{2}\right|\left|\sin\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right|. \end{aligned} \tag{6.38}$$

After these estimates, the lower bound of $E_{kl}^{GHVT} + E_{k'l'}^{GHVT}$ (following form the left-hand side inequality in (6.10)) is equal to minus the upper bound, and thus one can can apply the upper bound to the modulus of the left hand side of (6.16). This is because the only formal difference between expressions in the estimates seeking the lower bound of the averaged Eq. (6.10) compared to those seeking the upper bound boils down to the interchange between n_1 and n_2. After applying (6.36) and (6.37), this makes no difference anymore. One can shortly write

$$\left|E_{kl}^{GHVT}(\varphi_{kl}) + E_{k'l'}^{GHVT}(\varphi'_{k'l'})\right| \leq 4 - \frac{4}{\pi}\left|\sin\frac{\varphi_{kl}-\varphi'_{k'l'}}{2}\right| \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\sqrt{n_2^2+n_1^2}. \tag{6.39}$$

Going back to the magnitudes:

$$|E_{kl}^{GHVT}(\varphi_{kl}) + E_{k'l'}^{GHVT}(\varphi'_{k'l'})| \leq 4 - \frac{2\sqrt{2}}{\pi}\left|\sin\frac{\varphi_{kl}-\varphi'_{k'l'}}{2}\right| \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\sqrt{u_{kl}^2+v_{kl}^2}. \tag{6.40}$$

This inequality is valid for *any* choice of the plane of observables. The bound involves only the projections of vectors $\vec{u}$ and $\vec{v}$ onto the plane of the settings. The integrations in the bound can be thought of as a mean value of expression $\sqrt{u_{kl}^2+v_{kl}^2}$ averaged over the distribution of the vectors. For the plane orthogonal to the initial one the inequality (6.40), where u_{pq} and v_{pq} denote the projections of vectors $\vec{u}$ and $\vec{v}$, respectively, onto the plane spanned by the settings $\vec{a}_p$ and $\vec{b}_q$ (which is by construction orthogonal to the plane spanned by $\vec{a}_k$ and $\vec{b}_l$). We add the inequalities for orthogonal observatin planes,(6.40) and (6.41), and choose $\varphi'_{k'l'} = \varphi'_{p'q'} = 0$ and $\varphi_{kl} = \varphi_{pq} = \varphi$. This gives

$$\begin{gathered} \left|E_{kl}^{GHVT}(\varphi) + E_{k'l'}^{GHVT}(0)| + |E_{pq}^{\perp GHVT}(\varphi) + E_{p'q'}^{\perp GHVT}(0)\right|| \leq \\ 8 - \frac{2\sqrt{2}}{\pi}|\sin\frac{\varphi}{2}| \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\left(\sqrt{u_{kl}^2+v_{kl}^2}+\sqrt{u_{pq}^2+v_{pq}^2}\right) \end{gathered} \tag{6.41}$$

We apply the triangle inequality (6.33) to the expression within the bracket. This time vectors $\vec{x}$ and $\vec{y}$ have the following components:

$$\vec{x} = (u_{kl}, u_{pq}), \vec{y} = (v_{kl}, v_{pq}). \tag{6.42}$$

The integrand is bounded by:

$$\sqrt{u_{kl}^2+v_{kl}^2}+\sqrt{u_{pq}^2+v_{pq}^2} \geq \sqrt{(u_{kl}+u_{pq})^2+(v_{kl}+v_{pq})^2}. \tag{6.43}$$

Let us consider the term involving vector $\vec{u}$ only. Since the lengths are positive

$$(u_{kl}+u_{pq})^2 \geq u_{kl}^2 + u_{pq}^2. \tag{6.44}$$

Recall that u_{kl} and u_{pq} are projections onto orthogonal planes. One can introduce normal vectors to these planes, $\vec{n}_{kl}$ and $\vec{n}_{pq}$, respectively, and write

$$(\vec{n}_{kl}\cdot\vec{u})^2+u_{kl}^2=1,(\vec{n}_{pq}\cdot\vec{u})^2+u_{pq}^2=1. \tag{6.45}$$

Note that the scalar products are two components of vector $\vec{u}$ in the Cartesian frame build out of vectors $\vec{n}_{kl}$, $\vec{n}_{pq}$, and the one which is orthogonal to these two. Since vector $\vec{u}$ is normalized one has:

$$(\vec{n}_{kl}\cdot\vec{u})^2+(\vec{n}_{pq}\cdot\vec{u})^2\leq 1, \tag{6.46}$$

which implies for the sum of equations (6.45)

$$u_{kl}^2+u_{pq}^2\geq 1. \tag{6.47}$$

The same applies to vector $\vec{v}$ and one can conclude that

$$\sqrt{u_{kl}^2+v_{kl}^2}+\sqrt{u_{pq}^2+v_{pq}^2}\geq\sqrt{2}. \tag{6.48}$$

Since the weight function $F(\vec{u},\vec{v})$ is normalized, the final Laggett's tipe inequality is

$$\left|E_{kl}^{GHVT}(\varphi)+E_{k'l'}^{GHVT}(0)|+|E_{pq}^{\perp GHVT}(\varphi)+E_{p'q'}^{\perp GHVT}(0)\right|\leq 8-\tfrac{4}{\pi}\left|\sin\mathring{r}\tfrac{\varphi}{2}\right|. \tag{6.49}$$

Note that in contrast with canonical Leggett inequality (5.8) quantum theory cannot violate the inequality (6.48).

Conclusion

In this paper EPR paradox resolved successfully under new EPR nonlocality postulate. In order to resolve the EPR paradox [7] we apply a new quantum mechanical formalism based on the probability representation of continuous observables [8], [9], [10]. In addition in this paper the fundamental physics nature of the violation of the Bell inequalities is explained successfully under new EPR-B nonlocality postulate [8]. We show that the correlations of the observables involved in the Bohm-Bell type experiments can be expressed as correlations of classical random variables. The revised Bell type inequality in canonical notations reads $\langle AB\rangle+\langle A'B\rangle+\langle AB'\rangle-\langle A'B'\rangle\leq 6$. In contrast with canonical Bell inequalities the conventional quantum theory cannot violate the revisited Bell type inequalities.

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